Study materials of Mathematics for class D-III (H), Paper-VI on topic " Inner Product space"
Composed by Dr. Md. Shamim Ahmad Associate Professor, U.R. College Rosera (Samastipur)

Distance in an inner product space:

Theorem: Let u, v be any two vectors of inner product space. We define $d(u,v)$, the distance from u to v as

$$d(u,v) = \|u-v\|$$

It is to prove That

(i) $d(u,v) \geq 0$ and $d(u,v) = 0$ iff $u=v$

(ii) $d(u,v) = d(v,u)$

(iii) $d(u,v) \leq d(u,\omega) + d(\omega, v)$

(iv) $d(u,v) = d(u+\omega, v+\omega)$

Proof of (i) $d(u,v) = \|u-v\| > 0$ and $= 0$ iff $u-v=0 \Rightarrow u=v$

(ii) $d(u,v) = \|u-v\| = \|-(v-u)\| = |(-1)| \, \|v-u\|$
$= \|v-u\| = d(v,u)$

(iii) $d(u,v) = \|u-v\| = \|u-\omega+\omega-v\|$
$\leq \|u-\omega\| + \|\omega-v\|$
by Triangle inequality
$= d(u,\omega) + d(\omega, v)$

$\therefore d(u,v) \leq d(u,\omega) + d(\omega,v)$

(iv) $d(u,v) = \|u-v\| = \|(u+\omega)-(v+\omega)\|$
$= d(u+\omega, v+\omega)$

Hence d is a metric on the inner product space.

Example. If u and v be any two vectors in an inner product space, show That $\|u+v\|^2 - \|u-v\|^2 = 4\,\mathrm{Re}(u,v)$

Solution. We have

$$\|u+v\|^2 = (u+v,\, u+v)$$

$$= (u,\, u+v) + (v,\, u+v)$$

$$= (u,u) + (u,v) + (v,u) + (v,v)$$

$$= \|u\|^2 + (u,v) + (v,u) + \|v\|^2 \quad \text{—— (1)}$$

Also, $\|u-v\|^2 = (u-v,\, u-v)$

$$= (u,\, u-v) - (v,\, u-v)$$

$$= (u,u) - (u,v) - (v,u) + (v,v)$$

$$= \|u\|^2 - (u,v) - (v,u) + \|v\|^2 \quad \text{—— (2)}$$

Substracting (2) from (1), We get

$$\|u+v\|^2 + \|u-v\|^2 = 2(u,v) + 2(v,u)$$

$$= 2(u,v) + 2\overline{(u,v)}$$

$$= 2\left[(u,v) + \overline{(u,v)}\right]$$

$$= 2\cdot 2\,\mathrm{Re}(u,v)$$

$$= 4\,\mathrm{Re}(u,v)$$

Parallelogram law:

If u and v are vectors in an inner product space,

Prove that $\|u+v\|^2 + \|u-v\|^2 = 2\|u\|^2 + 2\|v\|^2$

Proof. We have $\|u+v\|^2 = (u+v, u+v)$

$$= (u, u+v) + (v, u+v)$$

$$= (u,u) + (u,v) + (v,u) + (v,v)$$

$$= \|u\|^2 + (u,v) + (v,u) + \|v\|^2 \quad \text{—— (1)}$$

Also, $\|u-v\|^2 = (u-v, u-v)$

$$= (u, u-v) - (v, u-v)$$

$$= (u,u) - (u,v) - (v,u) + (v,v)$$

$$= \|u\|^2 - (u,v) - (v,u) + \|v\|^2 \quad \text{—— (2)}$$

Adding (1) & (2), we have

$$\|u+v\|^2 + \|u-v\|^2 = 2\|u\|^2 + 2\|v\|^2$$

Proved.

Polarization Identity

If u and v are vectors in an inner product vector space

Prove that

$$4(u,v) = \|u+v\|^2 - \|u-v\|^2 + i\|u+iv\|^2 - i\|u-iv\|^2$$

Proof. Substracting (2) from (1), we have

$$\|u+v\|^2 - \|u-v\|^2 = 2(u,v) + 2(v,u) \quad \text{—— (3)}$$

Replacing v by iv in (3), we have

$$\|u+iv\|^2 - \|u-iv\|^2 = 2(u, iv) + 2(iv, u)$$

$$= 2\bar{i}(u,v) + 2i(v,u)$$

$$= -2i(u,v) + 2i(v,u) \quad \text{—— (4)}$$

Multiplying both sides of (4) by i, we get

$$i\|u+iv\|^2 - i\|u-iv\|^2 = -2i^2(u,v) + 2i^2(v,u)$$
$$= 2(u,v) - 2(v,u) \quad \text{—— (5)}$$

Adding (3) & (5), we get

$$\|u+v\|^2 - \|u-v\|^2 + i\|u+iv\|^2 - i\|u-iv\|^2 = 4(u,v)$$

Proved.

Orthogonality.

Orthogonal vectors:

Defn: Two vectors $u, v \in V$ where V is an inner product space, are said to be orthogonal if $(u,v) = 0$

In other words, $u \perp v \Rightarrow (u,v) = 0$

Vector orthogonal to a subset S of V

Let S be a subset of an inner product vector space V and let u be any vector in V.

Then u is said to be orthogonal to a set S i.e. $u \perp S$ i.e. u is orthogonal to every vector in S.

i.e. $u \perp v \ \forall v \in S$ i.e. $(u,v) = 0, \forall v \in S$

Hence $u \perp S \Rightarrow (u,v) = 0, \ \forall v \in S.$